

RUMOR PROCESSES ON \mathbb{N} AND DISCRETE RENEWAL PROCESSES

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ABSTRACT. We study two rumor processes on \mathbb{N} , the dynamics of which are related to an SI epidemic model with long range transmission. Both models start with one spreader at site 0 and ignorants at all the other sites of \mathbb{N} , but differ by the transmission mechanism. In one model, the spreaders transmit the information within a random distance on their right, and in the other the ignorants take the information from a spreader within a random distance on their left.

We obtain the probability of survival, information on the distribution of the range of the rumor and limit theorems for the proportion of spreaders. The key step of our proofs is to show that, in each model, the position of the spreaders on \mathbb{N} can be related to a suitably chosen discrete renewal process.

1. INTRODUCTION

In the last decades many works dealt with the analysis of the phenomenon of information transmission (news or rumors) from a probabilistic point of view. The resulting stochastic models are, in general, inspired in the classical SIR, SIS and SI epidemic models. In these models, it is assumed that an infection or information spreads through a population subdivided into susceptibles, infectives, and removed individuals, who are referred to as ignorants, spreaders, and stiflers when one deal with rumor diffusion processes.

In the context of the SIR epidemic model, rumor processes were introduced by Daley & Kendall (1965) and by Maki & Thompson (1973). In such models, a finite, closed and homogeneously mixing population is considered. Spreaders try to tell the rumor to ignorants, and stiflers appear either through the meeting of spreaders or through the meeting of a spreader with a stifler. The later transitions represent the loss of interest in propagating the rumor when a spreader meets someone that already knows the rumor. The well known results for these models are limit theorems for the remaining proportion of ignorants when there are no spreaders left in the population, that is, at the end of the process. Some generalizations of the basic models and recent results can be found, for instance, in Lebensztayn *et al.* (2011a,b), Comets *et al.* (2013), and references therein. Variations of the Maki-Thompson rumor model were considered in several graphs. For instance, the survival of the rumor was studied in Moreno *et al.* (2004) and Isham *et al.* (2010) when the population is represented by a random or complex network, and in Coletti *et al.* (2012) when the population is represented by the d -dimensional hypercubic lattice.

When the population is composed only by spreaders and ignorants, the rumor process is called SIS or SI epidemic model. In the former, a spreader may become ignorant, whereas in the later spreaders remain in such state forever. Recent results for the SIS model, known in the probabilistic literature as the contact process, which can be of interest in the context of information spreading, can be found in

Berger *et al.* (2005) and Durrett & Jung (2007). On the other hand, one of the first SI rumor models was proposed and analyzed, in an homogeneously mixing population, by Pittel (1987). In this case, the author studied the distribution of the number of stages before everybody is informed by means of an approximation on the number of spreaders at time t by a deterministic equation.

The purpose of this paper is to study rumor processes on \mathbb{N} , the dynamics of which are related to the process considered by Pittel (1987), but with long range transmissions. More precisely, we consider two long range rumor spreading models, initially introduced by Junior *et al.* (2011), called firework and reversed firework processes (FP and RFP in the sequel). Both models start with one spreader at site 0 and ignorants at all the other sites of \mathbb{N} . The difference between them is the transmission mechanism. In the FP, each spreader transmits the information, independently, to the individuals within a random distance on its right. In the RFP, each ignorant takes the information, independently, from a spreader within a random distance on its left.

Junior *et al.* (2011) gave sufficient conditions under which the rumor survives, or not, with positive probability. It is worth noticing that related results have been obtained recently by Bertacchi & Zucca (2013) and previously by Athreya *et al.* (2004) (the later in the context of space covering processes). In the present paper, we give necessary and sufficient conditions for survival of the rumor. Our method of proof, based on a direct comparison between the rumor processes and a discrete time renewal process, allows us to obtain several additional results. For the FP, we obtain the exact expression for the probability of survival. We also obtain information about the distribution of the range of the rumor when it dies out applying results of Garsia & Lamperti (1962), Bressaud *et al.* (1999) and Gallo *et al.* (2013), all of them concerning renewal theory. For the RFP, we obtain a law of large numbers and a central limit theorem for the proportion of spreaders in a range of size n as n diverges. We point out that this type of results, namely limit theorems for the proportion of individuals of a certain class, has been of interest in many of the papers previously cited. Related models, and results, can be found, for instance, in Kurtz *et al.* (2008), where information transmission is modeled by means of a system of random walks, or Andersson (1998), where an epidemic model is described in the framework of random graphs.

2. MODELS AND MAIN RESULTS

In what follows, $\mathbf{R} = (R_i)_{i \geq 1}$ will always be a sequence of \mathbb{N} -valued i.i.d. random variables with distribution

$$P(R_0 = k) = \lambda_k, \quad k = 0, 1, 2, \dots$$

where $\lambda_0 \in (0, 1)$. For each value of $k \geq 0$ let $\alpha_k := P(R_0 \leq k)$.

2.1. The Firework process. Suppose that one individual is disposed at each site of $\mathbb{N} = \{0, 1, \dots\}$. In this model, the spreaders transmit the information within a random distance to their right. For any $n \geq 0$, let A_n represents the set of individuals that have been informed at stage n in the Firework process. Initially, only 0 is a spreader, and thus $A_0 = \{0\}$. Then, the sequence $(A_n)_{n \geq 1}$ is defined recursively through

$$A_n := \{i \in \mathbb{N} : \text{there exists } j \in A_{n-1} \text{ such that } i \in \{j, \dots, j + R_j\}\} \setminus A_{n-1}.$$

In words, an individual is newly informed at stage n if it was an ignorant at stage $n - 1$, and if it was within the radius of transmission of a spreader on its left. Once informed, an ignorant becomes a spreader and remains a spreader forever. Then $\cup_{i \geq 0} A_i$ is the set of spreaders (or informed individuals) at the end of the spreading procedure (stage ∞). Let $M := |\cup_{i \geq 0} A_i|$ be the final number of spreaders. Note

that, in this case, M coincides with the final range of the rumor. The event “the rumor survives” writes as

$$\mathcal{A} := \{M = \infty\}.$$

Our first main result gives the exact probability of survival of the rumor.

Theorem 1.

$$P(\mathcal{A}) = \left(1 + \sum_{j \geq 1} \prod_{i=0}^{j-1} \alpha_i\right)^{-1}.$$

As a direct corollary, we see that survival occurs with positive probability if, and only if, $\sum_{j \geq 1} \prod_{i=0}^{j-1} \alpha_i < \infty$. An important issue that, as far as we know, has not been addressed in previous works concerning this model, is bounds to the tail distribution of the final range of the rumor. This is the object of the following results.

Proposition 1. *The random variable M has finite expectation when $\prod_{k \geq 0} \alpha_k > 0$, and has exponential tail distribution when α_k increases exponentially fast to 1.*

Under more specific assumptions, we can obtain more precise information on the tail distribution.

Proposition 2. *We have the following explicit bounds for the tail distributions.*

(i) *If $1 - \alpha_k \leq C_r r^k$, $k \geq 1$, for some $r \in (0, 1)$ and a constant $C_r \in (0, \log \frac{1}{r})$ then*

$$P(M \geq k) \leq \frac{1}{C_r} (e^{C_r r})^k.$$

(ii) *If $1 - \alpha_k \sim (\log k)^\beta k^{-\alpha}$, $\beta \in \mathbb{R}$, $\alpha > 1$, then there exists $C > 0$ such that, for large k 's, we have*

$$P(M \geq k) \leq C (\log k)^\beta k^{-\alpha}.$$

(iii) *If $1 - \alpha_k = \frac{r}{k}$, $k \geq 1$ where $r \in (0, 1)$, there exists $C > 0$ such that, for large k , we have*

$$P(M \geq k) \leq C \frac{(\ln k)^{3+r}}{(k)^{2-(1+r)^2}}.$$

(iv) *If $\alpha_k \sim ((k+1)/(k+2))^\alpha$, $\alpha \in (1/2, 1)$, then there exists $C = C(\alpha) > 0$ such that, for large k , we have*

$$P(M \geq k) \leq \frac{C}{k^{1-\alpha}}.$$

As examples, consider the following interesting variants of the model.

Example 1. *Instead of having exactly one individual at each site of \mathbb{N} , suppose that there is an individual at each site with probability $\epsilon \in [0, 1]$ independently of the other sites. The FP considered in Junior et al. (2011) corresponds to the particular case where $\epsilon = 1$. When $\epsilon < 1$, we obtain a rumor process in which the individuals are located at random positions and can be arbitrarily far away to any other individual. In this sense, ϵ is a “sparseness” parameter. This is a special case which is also studied by (Athreya et al., 2004, see Proposition 3.1 therein).*

In this variant, let $\bar{\mathbf{R}} = (\bar{R}_i)_{i \geq 1}$ be the i.i.d. sequence of random radius, and let $P(\bar{R}_0 = k) = \bar{\lambda}_k$ and $\bar{\alpha}_k := P(\bar{R}_0 \leq k)$. For any $i \geq 0$, let S_i be the Bernoulli random variable with parameter ϵ that indicates whether or not there is an individual at site i . Let us define

$$R_i := \bar{R}_i \cdot \mathbf{1}\{S_i = 1\}$$

which is the “effective” spreading radius of site i : if there is nobody at site i then the radius is 0, because no sites to the right of i are influenced by i . Otherwise, the radius is \bar{R}_i . Thus, all the results stated above hold using

$$\alpha_k = P(R_0 \leq k) = 1 - \epsilon(1 - \bar{\alpha}_k).$$

Example 2. Using different techniques Bertacchi & Zucca (2013) studied the following rumor processes in random environment. Consider $(X_n)_{n \geq 0}$ a sequence of \mathbb{N} -valued i.i.d. random variables and $(\bar{R}_n^i)_{i \geq 1, n \geq 1}$ a collection of i.i.d. random radius with $\bar{\alpha}_k = P(\bar{R}_n^i \leq k)$. At each site n , we have X_n individuals, and each individual i has a particular radius of spread \bar{R}_n^i .

In order to apply our results to this model, let us define

$$R_n := \sup_{i=1, \dots, X_n} \bar{R}_n^i. \quad (1)$$

We have $P(R_0 \leq k)$ equals

$$P\left(\sup_{i=1, \dots, X_0} \bar{R}_0^i \leq k\right) = \sum_{l \geq 1} P\left(\sup_{i=1, \dots, l} \bar{R}_0^i \leq k, X_0 = l\right) = \sum_{l \geq 1} P(X_0 = l) \bar{\alpha}_k^l.$$

where the last equality follows from the independence among all the r.v.'s involved. Our results hold using

$$\alpha_k = g_{X_0}(\bar{\alpha}_k)$$

where $g_{X_0}(\cdot)$ is the probability generating function of X_0 . In particular, Theorem 3.1 of Bertacchi & Zucca (2013) is a consequence of our Theorem 1. We further obtain tail decays for the size of the set of spreaders.

2.2. The Reverse Firework process. Similarly to the previous section, we suppose that there is one individual at each site of $\mathbb{N} = \{0, 1, \dots\}$. In this model, the ignorant individuals take the information of a spreader within a random distance on its left. We will now let B_n , $n \geq 0$ represents the set of individuals that have been informed at stage n in the Reverse Firework process. This sequence is also defined recursively through $B_0 = \{0\}$ and, for $n \geq 1$

$$B_n := \{i \in \mathbb{N} : \text{there exists } j \in B_{n-1} \text{ such that } j \in \{i - R_i, \dots, i\}\} \setminus B_{n-1}.$$

In words, an individual is newly informed at stage n if it was an ignorant at stage $n - 1$, and if its radius covers a spreader on its left. Once informed, an ignorant becomes a spreader and remains in that state forever. Then $\cup_{i \geq 0} B_i$ is the set of spreaders at the end of the spreading procedure (stage ∞) and we denote by N its cardinality. As in the FP we define the event “the rumor survives” by

$$\mathcal{B} := \{N = \infty\}.$$

For this model, we obtain necessary and sufficient conditions for survival of the rumor as well as the distribution of N when the rumor dies out.

Theorem 2. *There exist two situations.*

- If $\prod_{k \geq 0} \alpha_k = 0$, then $P(\mathcal{B}) = 1$.
- If $\prod_{k \geq 0} \alpha_k > 0$, then $P(\mathcal{B}) = 0$ and $N \sim \text{Geom}\left(\prod_{k \geq 0} \alpha_k\right)$.

For any $n \geq 1$, let $\zeta_n := \mathbf{1}\{n \in \cup_i B_i\}$, indicating whether the individual at site n is a spreader or not at the end of the procedure. Let also $N(n) := \sum_{i=1}^n \zeta_i$ denotes the number of spreaders in $\{1, \dots, n\}$. We will now state limit theorems for the proportion of spreaders within $\{1, \dots, n\}$, $N(n)/n$, when n diverges.

Let

$$\mu := 1 + \sum_{j \geq 1} \prod_{i=0}^{j-1} \alpha_i \quad \text{and} \quad \sigma^2 := \sum_{k \geq 1} k^2 (1 - \alpha_{k-1}) \prod_{i=0}^{k-2} \alpha_i - \mu^2. \quad (2)$$

Theorem 3. *If $\mu < \infty$ then*

$$\frac{N(n)}{n} \xrightarrow{\text{a.s.}} \mu^{-1},$$

and thus μ^{-1} is the final proportion of spreaders. Moreover, if $\sigma^2 \in (0, \infty)$, then

$$\sqrt{n} \left(\frac{N(n)}{n} - \mu^{-1} \right) \xrightarrow{\mathcal{D}} \mathcal{N} \left(0, \frac{\sigma^2}{\mu^3} \right).$$

Otherwise, $N(n)/n \rightarrow 0$.

In particular, observe that according to Theorems 2 and 3, if the α_k 's satisfy at the same time $\prod_k \alpha_k = 0$ and $\mu = \infty$ (for instance, if they are as in items (iii) and (iv) of Proposition 2), then the information reaches infinitely many individuals, but the final proportion of informed individuals is zero.

Example 3. Theorem 4.1 of Bertacchi & Zucca (2013) follows from Theorem 2 by considering R_k defined by (1). However, the distribution of N and the limit theorems for $N(n)$ are a novelty.

Example 4. Consider a model in which, if the nearest spreader on the left of site i is at distance k and $R_i \geq k$, then the individual at site i believes the information with probability p_k where p_k is a non-increasing sequence. Some examples:

- assuming $p_k = 1$ for any k , we retrieve the homogeneous case considered in Junior et al. (2011),
- assuming $p_k = \epsilon$ for any k , we obtain a model in which each individual is “susceptible” in that it believes the spreader within the radius on its left with a fixed probability ϵ ,
- assuming $p_k \searrow 0$, we convey the idea that the individual believes the nearest informed individual in the radius on its left with a probability which decreases according to its distance.

For this model, let $\bar{R} = (\bar{R}_i)_{i \in \mathbb{Z}}$ be the i.i.d. sequence of random radius, and let $P(\bar{R}_0 = k) := \bar{\lambda}_k$ and $\bar{\alpha}_k := P(\bar{R}_0 \leq k)$. This model is obtained as an example of the RFP, if we consider the i.i.d. sequence $(L_i)_{i \geq 1}$ with $P(L_i \leq k) = 1 - p_k$, $k \geq 0$ and we let

$$R_i = \min\{\bar{R}_i, L_i\}$$

denote the effective radius corresponding to this notion of susceptibility. In this case, Theorem 2 holds with

$$\alpha_k = 1 - p_k(1 - \bar{\alpha}_k).$$

3. THE DISCRETE TIME RENEWAL PROCESS

The proofs of our results will be based on a remarkable relationship between the rumor processes introduced in the preceding section and a specific discrete time renewal process. This relationship will be made explicit in Sections 4.1 and 4.2. The present section is dedicated to define the renewal process and list some of its properties. We will use, on purpose, the same notation as for the statements of the theorems.

Let $(q_k)_{k \geq 1}$ be a probability distribution on $\{1, 2, \dots\} \cup \{\infty\}$ defined by

$$q_k = (1 - \alpha_{k-1}) \prod_{i=0}^{k-2} \alpha_i,$$

and $q_\infty = 1 - \sum_k q_k$. Observe that, the mean and the variance of $(q_k)_{k \geq 1}$ are given by (2).

Let $\mathbf{T} = (T_n)_{n \geq 1}$ be an i.i.d. sequence of r.v.'s, taking values in $\{1, 2, \dots\} \cup \{\infty\}$ with common distribution $(q_k)_{k \geq 1}$. We call *discrete (undelayed) renewal process* the process $\mathbf{Y} = (Y_n)_{n \geq 0}$ defined through $Y_0 = 1$ and, for any $n \geq 1$, $Y_n = \mathbf{1}\{T_1 + \dots + T_i = n \text{ for some } i\}$. Observe that T_n is the distance between the $(n-1)^{\text{th}}$ and the n^{th} occurrence of 1 in \mathbf{Y} . As a consequence, $(q_k)_{k \geq 1}$ is called the *inter-arrival distribution*.

Each occurrence of an 1 is called a *renewal*. Let $u_n := \Pr(Y_n = 1)$, $n \geq 0$, be the corresponding discrete renewal sequence.

It is well-known that the chain \mathbf{Y} is recurrent if, and only if, $P(T = \infty) = \prod_{i \geq 0} \alpha_i = 0$ and, in the recurrent regime, it is positive recurrent if, and only if, $\mu < \infty$. The number of 1's (number of renewals) occurring in \mathbf{Y} up to time n , which we denote by $N(n)$, satisfies the following limit theorems (Ross, 2009, Chapter 7). If $\mu < \infty$, then $\frac{N(n)}{n} \xrightarrow{a.s.} \mu^{-1}$ and additionally, if $0 < \sigma^2 < \infty$, then

$$\frac{N(n) - n\mu^{-1}}{\sqrt{n\sigma^2/\mu^3}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1).$$

There are no simple explicit expressions for u_n , $n \geq 1$. The well-known Discrete Renewal Theorem (Ross, 2009, Chapter 7) states that $u_k \rightarrow \mu^{-1}$ and some results give information concerning the rate at which this convergence occurs. For instance, for the case where $\mu = \infty$, the following proposition is due to Bressaud *et al.* (1999).

Proposition 3. *When $\mu = \infty$, u_k converges to zero at*

- (i) *summable rate, if $\prod_{i \geq 0} \alpha_i > 0$ (that is, if $1 - \alpha_k$ summable);*
- (ii) *exponentially rate, if $1 - \alpha_k$ decreases exponentially to 0.*

The next proposition gives more explicit estimates under more specific assumptions. Items (i) and (iii) are due to Gallo *et al.* (2013, Proposition B.2), item (ii) is due to Bressaud *et al.* (1999, Remark 5) and item (iv) is due to Garsia & Lamperti (1962, Theorem 1.1).

Proposition 4. *We have the following explicit upper bounds.*

- (i) *If $1 - \alpha_k \leq C_r r^k$, $k \geq 1$, for some $r \in (0, 1)$ and a constant $C_r \in (0, \log \frac{1}{r})$ then*

$$u_k \leq \frac{1}{C_r} (e^{C_r r})^k.$$

- (ii) *If $\prod_{i \geq 0} \alpha_i > 0$ and $\sup_j \limsup_{k \rightarrow +\infty} (\frac{1 - \alpha_j}{1 - \alpha_{kj}})^{1/k} \leq 1$, then there exists a constant $C > 0$ such that, for large k , $u_k \leq C(1 - \alpha_k)$.*
- (iii) *If $\alpha_k = \frac{r}{k} + s_k$, $k \geq 1$ where $r \in (0, 1)$ and $\{s_n\}_{n \geq 1}$ is a summable sequence, there exists a constant $C > 0$ such that*

$$u_k \leq C \frac{(\ln k)^{3+r}}{(k)^{2-(1+r)^2}}.$$

- (iv) *If $\prod_{i \geq k+1} \alpha_i = L(k)k^{-\alpha}$ where $L(k) > 0$ and $\frac{L(\lambda k)}{L(k)} \rightarrow 1$ for any $\lambda > 0$ and $1/2 < \alpha < 1$, then, there exists $C(\alpha) > 0$ such that, for large k ,*

$$u_k \sim \frac{C(\alpha)}{k^{1-\alpha} L(k)}.$$

4. PROOFS

In this section, we construct the FP and the RFP through a sequence $\mathbf{U} = (U_n)_{n \in \mathbb{Z}}$ of iid r.v.'s uniformly distributed in $[0, 1[$. Let \mathbb{P} denotes the product law of \mathbf{U} . At each $i \geq 1$, U_i is used to specify the random radius

$$R_i := \sum_{k \geq 0} k \mathbf{1}\{\alpha_{k-1} \leq U_i < \alpha_k\} \text{ where } \alpha_{-1} := 0.$$

We recall that, for the FP process, this is the radius at which the individual at site i transmits the information on its right and, for the RFP, this is the radius at which the individual at site i takes the information on its left.

4.1. Firework Process. The proofs are based on the following lemma.

Lemma 1. *For any $n \geq 0$, we have $P(M > n) = u_{n+1}$.*

Proof. The first key point is to observe that M can be written as follows

$$M = \min\{i \geq 0 : R_j \leq i - j, j = 0, \dots, i\} \quad (3)$$

$$= \min\{i \geq 0 : U_j < \alpha_{i-j}, j = 0, \dots, i\}. \quad (4)$$

However, the proof will be simpler if we work with the *reversed* random variable

$$\bar{M} := \max\{i \leq 0 : U_j < \alpha_{j-i}, j = i, \dots, 0\},$$

which satisfies $-\bar{M} \stackrel{\mathcal{D}}{=} M$. Thus, what we have to prove is that $\mathbb{P}(\bar{M} < -n) = u_{n+1}$. The second key point is to observe that this definition of \bar{M} is similar to the definition of $\tau[0]$ considered in Comets *et al.* (2002) (see display (4.2) therein). The proof of our lemma would then, follow from a direct analogy with display (5.6) therein. We nevertheless include all the details here for completeness.

Let $(\mathbf{H}^{(m)})_{m \in \mathbb{Z}}$ be a family of Markov processes, where the index m indicates where each one starts, defined recursively using the single sequence \mathbf{U} as follows. For any $m \in \mathbb{Z}$, put $H_m^{(m)} = 0$ and

$$H_n^{(m)} = (H_{n-1}^{(m)} + 1) \mathbf{1}\{U_n < \alpha_{H_{n-1}^{(m)}}\}, \quad n > m.$$

The corresponding transition matrix Q is such that $Q(i, i+1) = \alpha_i$ and $Q(i, 0) = 1 - \alpha_i$. Since $\mathbf{H}^{(m)}$ is a Markov chain, it renews at each visit to 0 and the distance between two successive visits to 0 has distribution $q_k = (1 - \alpha_{k-1}) \prod_{i=0}^{k-2} \alpha_i$, where $\prod_{i=0}^{k-2} \alpha_i$ means that the chains climbs up from 0 to $k-1$, and $(1 - \alpha_{k-1})$ means that it falls down to 0. Consequently, for any $m \in \mathbb{Z}$ and $k \geq 1$, we have $\mathbb{P}(H_{m+k}^{(m)} = 0) = u_k$.

This family of coupled Markov processes has two important properties.

(1) *Monotonicity:*

$$H_n^{(m)} \geq H_n^{(k)}, \quad \forall m < k \leq n,$$

which implies in particular that $H_n^{(m)} = 0 \Rightarrow H_n^{(k)} = 0$ for all $m < k \leq n$.

(2) *Coalescence at 0*, that is

$$H_n^{(m)} = 0 \Rightarrow H_t^{(m)} = H_t^{(k)}, \quad \forall m \leq k \leq n \leq t.$$

Using these properties, we obtain the following sequence of equivalences, for any $j \leq 0$:

$$\begin{aligned} \bar{M} < -n &\Leftrightarrow \forall i \in \{-n, \dots, 0\}, \exists j \in \{i, \dots, 0\} : U_j > \alpha_{i-j} \\ &\Leftrightarrow \forall i \in \{-n, \dots, 0\}, \exists j \in \{i, \dots, 0\} : H_j^{(i-1)} = 0 \\ &\Leftrightarrow \forall i \in \{-n, \dots, 0\}, H_0^{(i-1)} = 0 \\ &\Leftrightarrow H_0^{(-n-1)} = 0, \end{aligned}$$

where the first line follows from the definition of \bar{M} , the second line follows from the definition of the family of Markov processes, the third line follows from the coalescing property, and the forth line follows from the monotonicity.

We therefore obtained that $\mathbb{P}(\bar{M} < -n) = \mathbb{P}(H_0^{(-n-1)} = 0) = \mathbb{P}(H_{n+1}^{(0)} = 0)$. Thus $\mathbb{P}(\bar{M} < -n) = u_{n+1}$. \square

Theorem 1 and Proposition 1 follow directly from Lemma 1, the fact that $u_k \rightarrow \mu^{-1}$ and Proposition 3. Proposition 2 follows from Proposition 4 by simple calculations.

4.2. Reversed Firework Process. Recall the definition of the sequence of the sets B_n , $n \geq 1$. For any $t, n \geq 1$, let $\zeta_n(t) := \mathbf{1}\{n \in \cup_{j \leq t} B_j\}$ and observe that $\zeta_n(t)$ is non-decreasing in t for each fixed $n \geq 1$. It follows that, by monotonicity, when t goes to infinity the sequence of processes $(\zeta(t))_{t \geq 1}$ converges weakly to the process ζ introduced in Section 2.2.

The proofs of the results are based on the following lemma.

Lemma 2. $\zeta \stackrel{D}{=} \mathbf{Y}$.

Proof of Lemma 2. For any sequence $a_m^n \in \{0, 1\}^{n-m+1}$, $-\infty \leq m \leq n < +\infty$, we define

$$\ell(a_m^n) := \inf\{i \geq 0 : a_{n-i} = 1\},$$

which is the number of zeros after the last occurrence of 1 in a_m^n . We use the convention that $\ell(a_m^n) = \infty$ when $a_i = 0$ for $i = m, \dots, n$. We have

$$\{\zeta_n = 1\} = \bigcup_{t \geq 1} \{\zeta_n(t) = 1\} = \bigcup_{t \geq 1} \{R_n \geq \ell(\zeta_0^{n-1}(t))\} = \{R_n \geq \ell(\zeta_0^{n-1})\} \quad (5)$$

where we used the fact that $\zeta_i(t)$ is non-decreasing in t in the first and in the last equalities. Observe that since $\zeta_0 = 1$, we always have $\ell(\zeta_0^{n-1}) \leq n-1$. Since U_n is independent of $\mathcal{F}(U_1^{n-1})$ with respect to which ζ_0^{n-1} is measurable, it follows that

$$\mathbb{P}(\zeta_n = 1 | \zeta_0^{n-1} = a_0^{n-1}) = \mathbb{P}(R_n \geq \ell(a_0^{n-1})) = 1 - \alpha_{\ell(a_0^{n-1})}.$$

In other words, in ζ , the conditional probabilities with respect to the “past” only depend on the distance until the last occurrence of an 1 (that is, the nearest occurrence of an 1 on the left). It follows that ζ is an 1 (at site 0) followed by a concatenation of i.i.d. blocks of random length K of the form $0^{K-1}1$. In other words, it is a renewal process with inter-arrival distribution $P(K = k)$, $k \geq 1$. Moreover, observe that $P(K = k) = (1 - \alpha_{k-1}) \prod_{i=0}^{k-2} \alpha_i$, which directly follows from

$$\mathbb{P}(\zeta_1^k = 0^{k-1}1 | \zeta_0 = 1) = \prod_{i=1}^k P(\zeta_i = 0 | \zeta_0^{i-1} = 10^{i-1}) \times \mathbb{P}(\zeta_k = 1 | \zeta_0^{k-1} = 10^{k-1}).$$

In other words, ζ is a renewal process with the same inter-arrival distribution as \mathbf{Y} , and thus, they have the same distribution as claimed. \square

The proof of Theorem 3 and of most of the statements of Theorem 2 follow directly from Lemma 2 and the results of Section 3 concerning \mathbf{Y} .

The only missing statement of Theorem 2 is that $N \sim \text{Geom}(r)$ when $r := \prod_k \alpha_k > 0$. But this can be seen from the fact that \mathbf{Y} renews at each visit to 1, and that at each visit, it has probability r of never coming back to 1.

5. DISCUSSION AND POSSIBLE EXTENSIONS

Here we list some interesting observation and possible extensions that are under consideration in works in progress.

- (1) Consider the FP and the RFP running with the same α_k 's. An interesting observation that may not be obvious at first glance is that the probability that the information reaches the individual at site n is equal in both processes. This is clear from the proofs (Lemmas 1 and 2).
- (2) A first natural extension is to consider the models where the individuals propagate or take the information on both sides (with same radii).
 - Obviously, for the FP, this change only makes sense if individuals are disposed on \mathbb{Z} , since on \mathbb{N} all the results are the same.

- For the RFP, the question still makes sense on \mathbb{N} . In fact, if the radius goes on both sides in the RFP on \mathbb{N} , conditions for survival are the same as in the original RFP, but results concerning the proportion of informed individuals will change.
- (3) Example 1 considers the case where the individuals are disposed according to an i.i.d. process (at each site, the probability that there is an individual is ϵ , independently of the other individuals). We see that the independence was crucial, as it allows us to compare each model with the original model with a new sequence of i.i.d. *effective radius*.
- Athreya *et al.* (2004) studied a case similar to Example 1, but where the sequence of individuals are disposed according to a Markov process. This case is not covered by Theorem 1. As they only obtain sufficient conditions for survival or not of the rumor, it is natural to wonder whether the results that we obtain in the i.i.d. case are also valid in the Markovian case.
 - A further natural extension would be to consider the case where the individuals are disposed according to an arbitrary renewal process. A similar extension could be studied for the RFP as well.

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